κ-Poincaré dispersion relations and the black hole radiation

A. Błaut*, J. Kowalski–Glikman† ‡, and D. Nowak–Szczepaniak§
Institute for Theoretical Physics
University of Wrocław
Pl. Maxa Borna 9
Pl–50-204 Wrocław, Poland

February 7, 2008

Abstract

Following the methods developed by Corley and Jacobson, we consider qualitatively the issue of Hawking radiation in the case when the dispersion relation is dictated by quantum κ-Poincaré algebra. This relation corresponds to field equations that are non-local in time, and, depending on the sign of the parameter κ, to sub- or superluminal signal propagation. We also derive the conserved inner product, that can be used to count modes, and therefore to obtain the spectrum of black hole radiation in this case.

*e-mail address ablaut@ift.uni.wroc.pl
†e-mail address jurekk@ift.uni.wroc.pl
‡Partially supported by the KBN grant 5PO3B05620
§e-mail address dobno@ift.uni.wroc.pl

1 Introduction

In the recent years there is a growing interest in investigations of the possible role played by modified (or broken) Lorentz invariance in ultra-high energy phenomena. There are many reasons for that. First of all we do not have an access yet to any experimental data concerning effects that take place at the energy scale close to Planck scale. Given this and the fact that there is something alarming in the ease one can, in principle, probe the Planck scale just by Lorentz boosting, it is natural to ask a question what would happen if one deforms the Lorentz (or Poincaré) symmetry. Moreover, it seems quite likely that we may have an access to the Planck scale physics in the near future, and perhaps, we already see traces of Planck-scale phenomena in the form of cosmic rays anomalies, that can be explained by making use of modification of Lorentz symmetry (see [1], [2] and references therein.)

Most of the papers studying the problem of modified Lorentz symmetry addressed the question as to how modified dispersion relation would influence physical phenomena, which might be "windows" to the Planck scale physics. One of them is the structure formation in inflationary cosmology, where the fluctuations that we see now in the form of temperature fluctuation in the background microwave radiation spectrum were initially to be of the size comparable with the Planck length [3], [4], [5], [6].

Another setting in which such analysis was performed is the black hole physics and the issue of Hawking radiation [8], [9], [10], [11]. This works has shown that the properties of Hawking radiation are highly insensitive to the class of deformations of dispersion relations, which has been considered. The only exception from this rule seems to be a case of a black hole with both inner and outer horizons considered in [12]. One should be careful however in making any judgment on the basis of a finite number of examples analyzed so far. The dispersion relation considered by Unruh [8] was devised so as to mimic a property of fluid, making possible observation of "sonic black holes". The dispersion relation of Corley and Jacobson, on the other hand can be understood only as a leading-order approximation of the unknown dispersion relation governing the Planck scale physics, and thus, their analysis cannot be regarded as fully conclusive.

In this paper we will repeat the analysis of Corley and Jacobson [10] in the case of dispersion relation motivated by a possible role that quantum algebras may play in fundamental physics.

The rest of this paper is organized as follows. In section 2 we introduce the κ-Poincaré dispersion relation and field equations in flat spacetime that follow from it, together with a properly defined inner product for complex scalar fields. Section 3 will be devoted to black hole radiation in the standard case, and in section 4 we will perform the qualitative analysis in the case of deformed dispersion relation.

2 Field equations in Minkowski spacetime

Our starting point will be the so-called κ-Poincaré algebra [13], [14], [15], being a quantum deformation of the standard Poincaré algebra that in recent months has become an object of intensive studies [16], [17], [18], [19], as a possible candidate for the algebra of *kinematical* symmetries of Planck scale physics. In the massless case this dispersion relation takes the form

$$\left(2\kappa\sinh\left(\frac{\omega}{2\kappa}\right)\right)^2 - \vec{k}^2 e^{\omega/\kappa} = 0, \tag{1}$$

where ω and $\vec{k}$ are energy and momentum, respectively, and κ is the parameter of dimension of mass (in the unit where c and $\hbar$ are set equal to 1), which is to be identified with the Planck mass[1]. At this point one should stress that there is an important difference between the case considered in this paper and the one analyzed by Unruh, Corley, Jacobson and others. Namely the dispersion relation (1) leads to field equations which are non-local in time, contrary to equations non-local in space studied before.

In the rest of the paper we will be dealing with the two-dimensional case, and in order to make the problem treatable, instead of using the dispersion relation (1) our starting point will be a slightly modified relation

$$k^2 = e^{-\omega/\kappa}\left(2\kappa\sinh\left(\frac{\omega}{2\kappa}\right)\right)^2 = \kappa^2\left(1-e^{-\omega/\kappa}\right)^2. \tag{2}$$

The major difference between these two relations is that (1) is *invariant* with respect to κ-Poincaré transformations (see, e.g., [15]), while (2) only covariant (i.e., invariant "on-shell", when (2) holds.)

In what follows we will consider two cases: $\kappa > 0$ which will be called super-luminal and $\kappa < 0$ which will be called sub-luminal. This terminology is motivated by the fact that in the former case the (momentum dependent) speed of massless modes, defined as $\mathcal{C} = \partial p_0/\partial p = \partial\omega/\partial k$ for $\omega > 0$, is greater than 1, and smaller than 1 in the latter (see [16] – [20]).

In order to compute a spectrum of black hole radiation one must have in disposal a conserved inner product, whose existence in the case of a system with infinite number of time derivatives is by no means clear, and moreover, even if such a product exists one must be able to explicitly find mutually orthogonal modes of positive and negative norms. Let us show therefore how such a product can be constructed in two dimensional Minkowski spacetime.

We start with the following equation of motion for the complex scalar field ϕ,

$$f(i\partial_t)\phi + \partial_x^2\phi = 0, \tag{3}$$

and its complex conjugate

$$f(i\partial_t)^*\phi^* + \partial_x^2\phi^* = 0, \tag{4}$$

[1] This identification is supported by the analysis of cosmic rays anomalies, see [1].

for some analytical function f, whose exact form will be given in (6) below. In order to construct the scalar product, we start with the following integral for the two arbitrary complex scalar fields ϕ_1, ϕ_2

$$\int_D dxdt \left\{ \phi_1^* \left(f(i\partial_t) + \partial_x^2 \right) \phi_2 - \phi_2 \left(f(i\partial_t) + \partial_x^2 \right)^* \phi_1^* \right\}$$

where D denotes a compact integration region, which is bounded by two space-like hypersurfaces at t_1 and t_2, say. The expression under integral is a divergence $\sim \partial_t Q + \partial_x J$. Now if we assume that the functions ϕ_i are on-shell and have a proper fall-off at spacial infinity, one can define the conserved inner product as

$$\Omega(\phi_1, \phi_2) = -i \int_{t=const} dx\, Q(\phi_1, \phi_2). \tag{5}$$

To find Q let us first expand

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad a_n \in \mathbb{R}. \tag{6}$$

Then it can be easily shown by partial integration that $Q(\phi_1, \phi_2)$ is of the form

$$Q(\phi_1, \phi_2) = \sum_{n=1}^{\infty} i^n a_n \sum_{k=0}^{n-1} (-1)^k \partial_t^k \phi_1^* \partial_t^{n-k-1} \phi_2. \tag{7}$$

Let us now show that plane waves solutions form an orthogonal system with respect to the product (5). Let

$$\phi_1 = e^{ik_1 x - i\omega_1(k_1)t}, \quad \phi_2 = e^{ik_2 x - i\omega_2(k_2)t}.$$

Then

$$\Omega(\phi_1, \phi_2) = 2\pi\delta(k_1 - k_2) \sum_{n=1}^{\infty} a_n \sum_{k=0}^{n-1} \omega_1^k \omega_2^{n-k-1}. \tag{8}$$

The norm of a plane wave is therefore

$$\Omega(\phi, \phi) \sim 2\pi \sum_{n=1}^{\infty} a_n \sum_{k=0}^{n-1} \omega^k \omega^{n-k-1} = 2\pi \sum_{n=1}^{\infty} n\, a_n\, \omega^{n-1} = 2\pi \frac{d\, f(\omega)}{d\, \omega}. \tag{9}$$

The waves with different k are orthogonal, of course. There is however one more case which must be considered, namely what happens when one has to do with a single k but two different ω. In the standard case, $\vec{k}^2 = \omega^2$ we have $\omega_1 = -\omega_2$, in the expansion (6) only $a_2 = 1$ differs from zero, and the situation is simple. In the case at hands, the reasoning is bit more involved. Since $\omega_1 \neq -\omega_2$ are related to the same k, using (6) one can write

$$k^2 = \sum_{n=0}^{\infty} a_n \omega_1^n = \sum_{n=0}^{\infty} a_n \omega_2^n,$$

from which we obtain

$$0=\sum_{n=0}^{\infty} a_n(\omega_1^n-\omega_2^n)=(\omega_1-\omega_2)\sum_{n=0}^{\infty} a_n\sum_{k=0}^{n-1}\omega_1^k\omega_2^{n-k-1}.$$

Thus we found that in general

$$\Omega(\phi_1,\phi_2)=2\pi\delta(k_1-k_2)\delta_{\omega_1,\,\omega_2}\frac{d\,f}{d\,\omega}(\omega_1), \tag{10}$$

and this enables us to find the pseudo-orthonormal basis for the inner product (5), which will be used in the analysis of the spectrum of the black hole radiation in the case of modified dispersion relation (2).

3 Hawking radiation, standard case

Here we recall briefly the step leading to derivation of Hawking radiation in the standard case, i.e., when $k^2=f(\omega)=\omega^2$.

Following Corley and Jacobson we consider the (two dimensional) black hole metric of the form

$$ds^2=dt^2-(dx-v(x)dt)^2, \tag{11}$$

where for the Schwarschild spacetime $v(x)=-\sqrt{2M/(x+2M)}$.

Consider a single frequency WKB mode of the form

$$\phi\sim\exp\left(i\int kdx\right)\,e^{-i\omega t}, \tag{12}$$

where ϕ is a massless Klein-Gordon field in the metric (11). Assuming that both k and v are slowly varying with position we get the dispersion relation of the form

$$(\omega-vk)^2=k^2, \tag{13}$$

from which $k=\omega/(1+v)$. Observe that this equation defines a frequency $\omega'=\omega-vk$ being the frequency of the wave as seen by the freely falling observer. We seek the minimal (negative) value of v for which eq. (13) has a solution. In the case at hands we take $v=-1$ and $x=0$ to correspond to the horizon of black hole. Now we can expand around this point to get $v\simeq-1+\varkappa\,x$, where $\varkappa=v'(0)=1/4M$ is the surface gravity. Inserting this to equation (12) we find that

$$\phi\sim\exp\left(i\frac{\omega}{\varkappa}\,\log(x)\right). \tag{14}$$

To extract the positive and negative frequency parts as defined by freely falling observers near the horizon, we must analytically continue the solution to negative x through the upper and lower complex x plane. In this way we obtain

two functions ϕ_+ and ϕ_- corresponding to positive and negative wavevectors, respectively:

$$\phi_+ = \phi + \exp{(-\pi \frac{\omega}{\varkappa})} \tilde{\phi}, \quad \phi_- = \phi + \exp{(\pi \frac{\omega}{\varkappa})} \tilde{\phi}, \tag{15}$$

where $\tilde{\phi}$ is defined by

$$\tilde{\phi}(x) = \begin{cases} 0 & x > 0 \\ \phi(-x) & x < 0. \end{cases} \tag{16}$$

They agree with (14) on the positive x axis and their ratio for $x < 0$ equals

$$\frac{\phi_+}{\phi_-} = \exp\left(-2\pi \frac{\omega}{\varkappa}\right). \tag{17}$$

To compute the average number of particles of energy $\hbar\omega$ produced in the Hawking process one has to evaluate the square of the norm of the negative frequency part of the resulting initial mode. The positive frequency final mode vanishing for the negative x is equal

$$\phi \ \sim \psi = \phi_+ - \exp{(-2\pi \frac{\omega}{\varkappa})} \phi_-. \tag{18}$$

Using the fact that Klein-Gordon inner product Ω^{KG} satisfies

$$\Omega^{KG}(\phi_+, \phi_+) = 1 - \exp{(-2\pi \frac{\omega}{\varkappa})}, \quad \Omega^{KG}(\phi_-, \phi_-) = 1 - \exp{(2\pi \frac{\omega}{\varkappa})},$$
$$\Omega^{KG}(\phi_+, \phi_-) = 0, \tag{19}$$

one finds

$$- < n_\omega >= \frac{\Omega^{KG}(\phi_-, \phi_-) \exp{(-4\pi \frac{\omega}{\varkappa})}}{\Omega^{KG}(\psi, \psi)} =$$
$$= \frac{1}{\frac{\Omega^{KG}(\phi_+, \phi_+)}{\Omega^{KG}(\phi_-, \phi_-)} \exp{(4\pi \frac{\omega}{\varkappa})} + 1} = \frac{1}{-\exp{(2\pi \frac{\omega}{\varkappa})} + 1} \tag{20}$$

which is exactly the Hawking formula.

4 Black hole radiation with κ-Poincare dispersion

In the metric (11) the field equation describing the dynamics of massless scalar field is given by

$$\hat{S}\phi \equiv \left[\kappa^2 (1 - e^{-\frac{i}{\kappa}(\partial_t + \partial_x \ v(x))})(1 - e^{-\frac{i}{\kappa}(\partial_t + v(x) \ \partial_x)}) + \partial_x^2\right] \phi = 0. \tag{21}$$

Because $\hat{S}$ is a self–adjoint differential operator in the sense that

$$\int dt \, dx \, \psi_1^* \hat{S} \psi_2 = \int dt \, dx \, (\hat{S}\psi_1)^* \psi_2 \tag{22}$$

for complex functions ψ_1, ψ_2 vanishing with all derivatives at the boundary of the integration domain, it is justified to proceed with construction of the inner product as it was done in Section 2.

Now we again make use of the WKB approximation (12) and substitute it into eq. (21) neglecting the terms $\partial_x v$ and $\partial_x k/k$. As the result we obtain

$$k^2 = \kappa^2(1 - e^{-(\omega - v(x)k)/\kappa})^2 \equiv f(\omega - v(x)k) = f(\omega'). \tag{23}$$

Using this and equation (9), one sees that in regions where v is approximately constant, solutions satisfying $i(\partial_t + v\partial_x)\phi = \omega'\phi$ have positive (negative) Klein–Gordon norm for $\omega' > 0$ ($\omega' < 0$).

The intercept points of the line $\omega' = \omega - v(x)k$ with x-dependent slope and the curve $k^2 - f^2(\omega') = 0$ on the (k, ω') plane correspond to possible values of wavevectors. In the case of κ-Poincaré dispersion relation (2), we have

$$\omega' = -\kappa \log\left(1 \pm \frac{|k|}{\kappa}\right) \equiv F(k) \tag{24}$$

for super-luminal $\kappa > 0$, and for the sub-luminal, $\kappa < 0$ cases (see Figures 1 and 2 for details.)

Let us pause at this point to make an important comment. Any reasonable function $F(k)$ must satisfy the condition

$$F(k) \sim k$$

for sufficiently small k, so that it corresponds to the standard dispersion relation for small momenta. To make this statement more precise, it should be noted that from purely dimensional reason, the function $F(k)$ must contain a scale, so it is of the form $F(k;\kappa)$ and satisfies the condition

$$F(k;\kappa) = k + O\left(\frac{k^2}{\kappa^2}\right) \quad \text{for } k/\kappa \ll 1.$$

This means that for ω small enough (i.e., $\omega/\kappa \ll 1$) the solutions of eq. (23) is in leading order the same as in the standard case. One can conclude therefore that, if one defines the initial vacuum as seen by the freely falling observer near the horizon, for low frequencies the spectrum will be almost thermal with differences in the high frequency part of the spectrum. If the qualitative thermal behavior holds for high frequencies as well, then the modifications due to the deformed dispersion relation would be exponentially suppressed, at least for temperatures (surface gravity) small compared to the scale κ. But, of course, it does not make sense at all to consider situation when the temperature is of order of, or higher than κ, because such a regime corresponds to quantum gravity (recall that κ is of order of Planck scale) and the approximation of Schwarzschild background geometry would almost certainly not hold.

The question then arises if for temperatures reasonably below κ scale there is any deviation from the Hawking result[2]. It should be stressed that the rationale for asking this question is rather different from that in [8] – [12]. There the question was if, by making use of a non-standard dispersion relation, one can avoid trans-Planckian frequencies keeping at the same time the qualitative picture of Hawking process. Here our goal is different: we have the dispersion relation to start with, and the question we ask ourselves is if it does change the thermal behavior of black holes?

To answer this question consider the group velocity of a wavepacket. It can be expressed as

$$\upsilon_g = \upsilon'_g + v(x), \tag{25}$$

where $\upsilon_g = d\omega/dk$ is the group velocity with respect to the static frame and $\upsilon'_g = d\omega'/dk$ is the one corresponding to the freely falling frame. The detailed discussion presented in [10] indicates that the *sin equa non* condition for Hawking radiation to occur is that there exists an ingoing mode (i.e., a mode with $\upsilon_g < 0$) with *negative* free-fall frequency ω' for positive Killing frequency ω outside the horizon (i.e., for $v(x) > -1$).

Knowing this we can turn to the analysis of the κ-Poincaré dispersion relation. Let us consider first the super-luminal case ($\kappa > 0$). The corresponding picture can be found in Figure 1.

We see that there are three intersections of the line $\omega - v(x)k$ with the curve $F(k)$, eq. (24) which, following notation of [10], we call (from the left to the right of the figure) k_-, k_{-s}, k_+. For these intersections one can estimate the signs of group velocities in both frames of reference

$$\begin{array}{llll} \text{for } k_- & \upsilon'_g > 0 & \upsilon_g < 0 & \rightarrow \text{ ingoing packet} \\ \text{for } k_{-s} & \upsilon'_g < 0 & \upsilon_g < 0 & \rightarrow \text{ ingoing packet} \\ \text{for } k_+ & \upsilon'_g > 0 & \upsilon_g > 0 & \rightarrow \text{ outgoing packet.} \end{array} \tag{26}$$

Following the standard analysis (see also [10]) we conclude that the number of created particles in final state ψ_{out} is given by

$$n(\psi_{out}) = -\Omega(\psi_-, \psi_-), \tag{27}$$

where ψ_- denotes the ingoing part of the solution with negative free-fall frequency. In our case the solution corresponding to the wavevector k_- satisfies this condition. However we do not know whether it is possible for the outgoing solution centered around k_+ to undergo the mode conversion which is essential in the analysis. Although the naive diagram analysis indicates that it is not the case, one must remember that near the horizon the WKB approximation brakes down and the mode conversion as well as the particle production might be possible. Here, unlike in [11] and [12], the negative frequency modes that give rise

[2] At this point it is worth recalling that the similar question has been asked in the context of inflationary cosmology (i.e., is it any deviation from Harrison–Zeldovich spectrum if one makes use of modified dispersion relation [3]–[7]), and it turned out that the answer depends on the form of relation used, as well as the form of initial conditions.

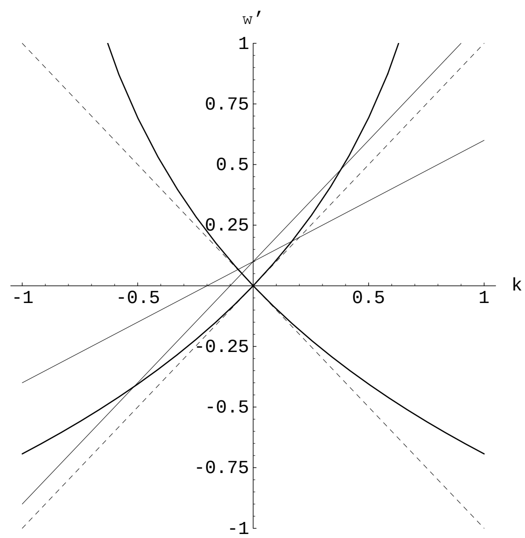

Figure 1: The behavior ω' vs. k in super-luminal case ($\kappa = 1$).

to particle production originate at infinity rather than inside the horizon. Still, in order to reach any final conclusion, one needs to perform detailed numerical studies.

Let us now turn to the sub-luminal case, illustrated in Figure 2. In this case for given positive ω outside the horizon there are, depending on the distance from the horizon, either three solutions k_{-s}, k_{+s}, k_{+} (ingoing, outgoing, ingoing mode respectively), or two k_{-s}, k_{+s} (ingoing, hanging), or only one k_{-s}. None of them contributes to the creation of particles since all have a positive free-fall frequency. Again this conclusion is drawn under the assumption of the validity of WKB approximation which for small Killing frequencies ω brakes in the vicinity of the horizon. Taking sufficiently small ω one can get arbitrarily close to the horizon. Then the mode conversion to negative ω' might be possible resulting in two negative wavevector wavepackets, one of which would represent the ingoing mode. Such negative frequency wavepacket giving rise to the black hole radiation would originate far inside the horizon. Note, however, that there is a difference between the subluminal dispersion case considered by [10] and the one discussed here. The intersection with the negative branch of dispersion relation occurs only inside the horizon. The appearance of a "gap" makes the mode conversion less obvious.

In the analysis above we followed exactly the methods developed by Corley and Jacobson [10], but this is only half of a story. In our case dispersion relations are not symmetric with respect to inversion $\omega' \to -\omega'$, and therefore we have to

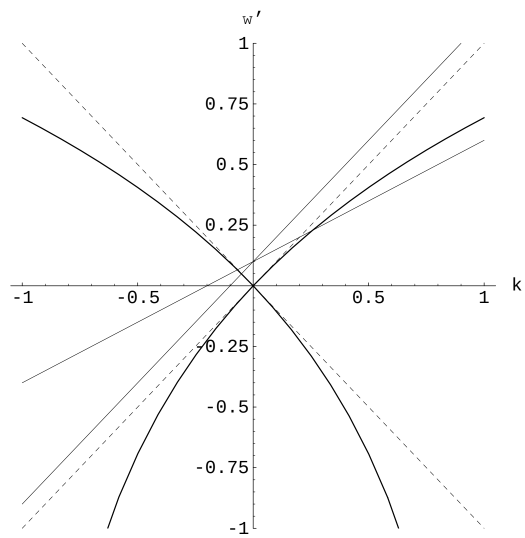

Figure 2: The behavior ω' vs. k in sub-luminal case ($\kappa = -1$).

do with complex field equations. Since the corresponding field operator $\hat{\phi}$ is not hermitian one might, in analogy with the standard analysis, define creation and annihilation operators for particles and antiparticles, the latter corresponding to negative frequency waves.

Therefore the complete analysis of the spontaneous creation of particles (see for example [22], [23]) should include tracing both the outgoing positive frequency and outgoing negative frequency modes backwards in time, and will be presented elsewhere [21].

We have shown therefore that in both cases the existence of "Hawking-type" radiation is possible. However, in both super- and subluminal regimes there are obstacles that, in principle, may lead to strong suppression of the radiation for higher frequencies. Of course both these results rely on the validity of WKB regime, and the detailed analysis of this regime as well as the quantitative numerical study of the qualitative results of this paper will be presented elsewhere [24]. The analysis presented here indicates however that such studies are worth undertaking, since the qualitative picture which emerged from our investigations presented here differs in many respects from the one studied before.

Acknowledgement. We would like to thank Ted Jacobson for discussion and comments.

References

[1] G. Amelino-Camelia and T. Piran, Phys.Rev. D64 (2001) 036005, `astro-ph/0008107` .

[2] G. Amelino-Camelia, J. Lukierski and A. Nowicki `hep-th/0103227`.

[3] Jérôme Martin and Robert H. Brandenberger, Phys. Rev., **D63**, 123501 (2001); Robert H. Brandenberger and Jérôme Martin `astr-ph/0005432`.

[4] J. Kowalski-Glikman, Phys. Lett. **B 499**, 1 (2001).

[5] J.C. Niemeyer, Phys. Rev., **D63**, 123502 (2001).

[6] A.A. Starobinsky Pisma Zh.Eksp.Teor.Fiz. **73**, 415 (2001).

[7] J.C. Niemeyer and R. Parentani, `astro-ph/0101451`.

[8] W.G. Unruh, Phys. Rev. **D 51**, 2827 (1995).

[9] R. Brout, S. Massar, R. Parentani, and P. Spindel, Phys. Rev. **D 52**, 4559 (1995).

[10] S. Corley and T. Jacobson, Phys. Rev. **D 54**, 1568 (1996).

[11] S. Corley, Phys. Rev. **D 57**, 6280 (1998).

[12] S. Corley and T. Jacobson, Phys. Rev. **D 59**, 124011 (1999).

[13] J. Lukierski, A. Nowicki, H. Ruegg and V.N. Tolstoy, Phys. Lett. **B264**, 331 (1991).

[14] S. Majid and H. Ruegg, Phys. Lett. **B334**, 348 (1994).

[15] J. Lukierski, H. Ruegg and W.J. Zakrzewski, Ann. Phys. **243**, 90 (1995).

[16] G. Amelino-Camelia, `gr-qc/0012051`.

[17] G. Amelino-Camelia, Phys.Lett. B510 (2001) 255-263 ,`hep-th/0012238`.

[18] J. Kowalski-Glikman, `hep-th/0102098`.

[19] G. Amelino-Camelia, `gr-qc/0106004`, Proceedings of the 37th Karpacz Winter School on Theoretical Physics, to appear.

[20] N.R. Bruno, G. Amelino-Camelia, and J. Kowalski-Glikman, `hep-th/0107039`.

[21] A.Blaut in preparation

[22] G. Gibbons, Comm. Math. Phys. **44**, 245 (1975)

[23] R.M. Wald, Ann. Phys. **118**, 490 (1979)

[24] D. Nowak-Szczepaniak, in preparation.